Intro GGT Exercises 4

General suggestion: draw some slim triangles.

Exercise 1 (Gromov Lemma 7.2.C, "discrete" local-to-global principal of quasi-geodesics)**.**
Let (X, d) be a δ-hyperbolic space. In this exercise we denote $d(x, y) = |x - y|$ for convenience.
Let $(x_i)_{0 \le i \le n}$ be a sequence of points in X, and let $c > 0$. Assume that for every $1 \le i \le n - 1$,

$$|x_{i+1} - x_{i-1}| \ge \max\{|x_{i+1} - x_i|,\ |x_i - x_{i-1}|\} + 2\delta + c. \tag{1}$$

We shall prove that

$$|x_n - x_0| \ge cn.$$

1. By the hyperbolicity and (1), show that

$$2(x_{i-1}, x_{i+1})_{x_i} \le |x_{i-1} - x_i| - 2\delta - c. \tag{2}$$

2. Show that

$$(x_{i-2}, x_i)_{x_{i-1}} + (x_{i-1}, x_{i+1})_{x_i} \le |x_{i-1} - x_i| - 2\delta - c. \tag{3}$$

We shall prove by induction on i that for all $1 \le i \le n$,

$$(x_{i-2}, x_i)_{x_{i-1}} \ge (x_0, x_i)_{x_{i-1}} - \delta, \tag{4}$$

and

$$|x_0 - x_i| \ge ic. \tag{5}$$

3. Verify the base cases $i = 1$ and $i = 2$.

4. Assume the induction hypothesis holds for $i = n - 1$. Use the identity

$$(x_{n-2}, x_n)_{x_{n-1}} = |x_{n-1} - x_{n-2}| - (x_0, x_{n-1})_{x_{n-2}},$$

   (4) with $i = n - 1$, and (3), show that

$$(x_0, x_{n-2})_{x_{n-1}} \ge (x_{n-2}, x_n)_{x_{n-1}} + \delta + c.$$

5. By the fact (δ-hyperbolicity) that

$$(x_{n-2}, x_n)_{x_{n-1}} \ge \min\{(x_0, x_{n-2})_{x_{n-1}}, (x_0, x_n)_{x_{n-1}}\} - \delta,$$

   deduce that we have (4) with $i = n$. That is,

$$(x_{n-2}, x_n)_{x_{n-1}} \ge (x_0, x_n)_{x_{n-1}} - \delta. \tag{6}$$

6. Use the identity

$$|x_0 - x_n| = |x_0 - x_{n-1}| + |x_{n-1} - x_n| - 2(x_0, x_n)_{x_{n-1}}$$

   together with (4) with $i = n$, (5) with $i = n - 1$, and (2), prove that

$$|x_0 - x_n| \ge nc.$$

Exercise 2 (Christmas tree lemma)**.** Let (X, d) be a δ-hyperbolic geodesic metric space. We shall prove that for any path $\gamma : [0, \ell] \to X$, every point $x \in X$ on a geodesic $[\gamma(0), \gamma(\ell)]$ satisfies

$$d(x, \gamma) \leq 4\delta \log_2 \left(\frac{\ell}{\delta}\right) + \delta.$$

First, we prove by induction that for any integer $N \geq 0$ and any path $\gamma : [0, \ell] \to X$, every point x on a geodesic $[\gamma(0), \gamma(\ell)]$ satisfies

$$d(x, \gamma) \leq 4\delta N + \frac{\ell}{2^{N+1}}.$$

1. Verify the base case $N = 0$.

In the following, assume that the inequality holds for $N - 1$ and for any path.

2. Consider a geodesic triangle $[\gamma(0), \gamma(\ell/2), \gamma(\ell)]$. Justify that for any $x \in [\gamma(0), \gamma(\ell)]$, there exists a point
$$x' \in [\gamma(0), \gamma(\ell/2)] \cup [\gamma(\ell/2), \gamma(\ell)]$$
such that
$$d(x, x') \leq 4\delta.$$

3. Assume for instance that $x' \in [\gamma(0), \gamma(\ell/2)]$ (the other case is symmetric). Estimate $d(x', \gamma')$ by applying the induction hypothesis to the subpath $\gamma' = \gamma|_{[0,\ell/2]}$ and the point x'.

4. Deduce that
$$d(x, \gamma) \leq 4\delta N + \frac{\ell}{2^{N+1}}.$$

5. Now let
$$N = \left\lfloor \log_2 \left(\frac{\ell}{\delta}\right) \right\rfloor.$$
Show that
$$d(x, \gamma) \leq 4\delta \log_2 \left(\frac{\ell}{\delta}\right) + \delta.$$